Spring'21 Discrete Math Qual. Solutions

1) (20 points): 10 pts each for a, b. (Chapter 3, Section 3.3)

$A_i = \{\ldots, -2, -1, 0, 1, \ldots, i\}$

a)

$$\bigcup_{i=1}^{n} A_i = A_1 \cup A_2 \cup A_3 \cup \cdots \cup A_n$$

$$A_1 = \{\ldots, -2, -1, 0, 1\}$$
$$A_2 = \{\ldots, -2, -1, 0, 1, 2\}$$
$$A_3 = \{\ldots, -2, -1, 0, 1, 2, 3\}$$
$$\vdots \qquad \vdots \qquad \vdots \qquad \vdots$$
$$A_n = \{\ldots, -2, -1, 0, 1, 2, \ldots, n\}$$

So, $\bigcup_{i=1}^{n} A_i = A_n$

b)

$$\bigcap_{i=1}^{n} A_i = A_1 \cap A_2 \cap A_3 \cdots \cap A_n$$

$A_1 \ldots A_n$ sets are as given above.

So, $\bigcap_{i=1}^{n} A_i = \{\ldots, -2, -1, 0, 1\}$

2) (30 points): 5 for gcd, 10 for Ext. Euclidean, 5 for 1st inverse, 5 for 2nd inverse. (Ch. 9, Sec. 9.5)

First, we find gcd using Euclidean algorithm.

Let $a = 78$, $b = 35$

$q_1 = 2$	$78 \bmod 35 = 8$
$q_2 = 4$	$35 \bmod 8 = 3$
$q_3 = 2$	$8 \bmod 3 = 2$
$q_4 = 1$	$3 \bmod 2 = 1 \longrightarrow$ gcd
$q_5 = 2$	$2 \bmod 1 = 0$

Second, we use Extended Euclidean algorithm to find $x, y \in \mathbb{Z}$, s.t.

$$\underline{1} = x \cdot 78 + y \cdot 35$$

$$\begin{aligned}
1 &= 3 \bmod 2 \\
&= 3 - 2 \\
&= 3 - (8 \bmod 3) \\
&= 3 - (8 - 3(2)) \\
&= 3(1) - 8 + 3(2) \\
&= 3(3) - 8 \\
&= 3(35 \bmod 8) - 8 \\
&= 3(35 - 8 \cdot 4) - 8 \\
&= 3(35) - 8(12) - 8(1) \\
&= 3(35) - 13(8) \\
&= 3(35) - 13(78 \bmod 35)
\end{aligned}$$

$$= 3(35) - 13\,(78 - 35(2))$$
$$= 3\,(35) - 13\,(78) + 35\,(26)$$
$$= 29 \cdot 35 - 13 \cdot 78$$

So, $1 = 29 . 35 - 13 . 78$

$x = -13,\ y = 29$

Multiplicative inverse of $(35 \bmod 78)$:
Coefficient of 35 is 29. So, 35^{-1} is 29

Sanity check: $(35 \cdot 29) \bmod 78 \stackrel{?}{=} 1$ ✓

Multiplicative inverse of $(78 \bmod 35)$:
Coefficient of 78 is -13. We need to find $(-15 \bmod 35)$
From fundamental theorem of division:

$$-13 = 35 \cdot q + r \qquad (q = \text{quotient},\ r = \text{remainder})$$
So $q = -1,\ r = 22 \quad (0 \leq r < 35,\ r \in \mathbb{Z}^{+})$

Sanity check: $(78 \cdot 22) \bmod 35 \stackrel{?}{=} 1$ ✓

3) (25 points). Let $P(n)$ denote the assertion:

$$\sum_{j=0}^{n} (-1/2)^j = \frac{2^{n+1} + (-1)^n}{3 \cdot 2^n} \quad ; \quad n \geq 0$$

Proof:

1) Base case: $n=1$. LHS $= \sum_{j=0}^{1} \left(-\frac{1}{2}\right)^j = \left(\frac{-1}{2}\right)^0 + \left(\frac{-1}{2}\right)^1$

$= 1 - \frac{1}{2} = \frac{1}{2}$

RHS $= \frac{2^2 + (-1)^1}{3 \cdot 2^1} = \frac{4-1}{6} = \frac{1}{2}$

So, $P(1)$ is true.

2) Inductive Hypothesis (I.H.): Assume $P(k)$ is true

i.e., $\sum_{j=0}^{k} \left(-\frac{1}{2}\right)^j = \frac{2^{k+1} + (-1)^k}{3 \cdot 2^k}$

3) Induction: We need to show $P(k+1)$ is true, i.e.,

$$\sum_{j=0}^{k+1} \left(-\frac{1}{2}\right)^j = \frac{2^{k+2} + (-1)^{k+1}}{3 \cdot 2^{k+1}}$$

$$\text{LHS} = \sum_{j=0}^{k+1} \left(-\frac{1}{2}\right)^j = \sum_{j=0}^{k} \left(-\frac{1}{2}\right)^j + \left(\frac{-1}{2}\right)^{k+1}$$

$$= \frac{2^{k+1} + (-1)^k}{3 \cdot 2^k} + \left(\frac{-1}{2}\right)^{k+1} \quad \text{(from I.H.)}$$

$$= \frac{2^{k+1} + (-1)^k}{3 \cdot 2^k} + \frac{(-1)^k \cdot (-1)^1}{2^k \cdot 2^1}$$

$$= \frac{1}{2^k}\left[\frac{2 \cdot (2^{k+1} + (-1)^k) - 3 \cdot (-1)^k}{3 \cdot 2}\right]$$

$$= \frac{1}{2^k}\left[\frac{2^{k+2} + 2(-1)^k - 3(-1)^k}{3 \cdot 2}\right]$$

$$= \frac{1}{2^k}\left[\frac{2^{k+2} - (-1)^k}{3 \cdot 2}\right]$$

Now, we know $(-1)^k = -(-1)^{k+1}$, e.g.: $(-1)^2 = 1$, $(-1)^3 = -1$, $(-1)^4 = 1$, and so on.

So, $LHS = \frac{2^{k+2} + (-1)^{k+1}}{3 \cdot 2^{k+1}}$

$= RHS$

So, $\sum_{j=0}^{k+1} (-1/2)^j = \frac{2^{k+2} + (-1)^{k+1}}{3 \cdot 2^{k+1}}$

(Ch. 8, Sec. 8.4, 8.5)

QED

4) 25 points : 5 each (can leave answers as expression, no need to find exact value)

a) 26^8

b) $P(26,8) = \dfrac{26!}{18!} = 26 \cdot 25 \cdot 24 \cdot 23 \cdot 22 \cdot 21 \cdot 20 \cdot 19$

c) $1 \cdot 1 \cdot 26^6 = 26^6$

d) $1 \cdot 1 \cdot 1 \cdot 1 \cdot 26^4 = 26^4$

e) $\underset{\text{(start only)}}{1 \cdot 1 \cdot 26^6} + \underset{\text{(start \& end, both)}}{1 \cdot 1 \cdot 1 \cdot 1 \cdot 26^4} + \underset{\text{(end only)}}{1 \cdot 1 \cdot 26^6}$

$= 26^6 + 26^4 + 26^6$ (Ch. 10, Sec. 10.1, 10.3)